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DERIVED LENGTH AND PRODUCTS OF CONJUGACY CLASSES

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ABSTRACT

Let G be a supersolvable group and A be a conjugacy class of G. Observe that for some integer $\eta(AA^{-1}) > 0$, $AA^{-1} = \{ab^{-1} \colon a, b \in A\}$ is the union of $\eta(AA^{-1})$ distinct conjugacy classes of G. Set $\mathbf{C}_G(A) = \{g \in G \colon a^g = a \text{ for all } a \in A\}$. Then the derived length of $G/\mathbf{C}_G(A)$ is less or equal than $2\eta(AA^{-1}) - 1$.

1. Introduction

Let G be a finite group, A be a conjugacy class of G and e be the identity of G. Let X be a nonempty G-invariant subset of G, i.e. $X^g = \{g^{-1}xg \colon x \in X\} = X$ for all $g \in G$. Then for some integer $n > 0$, X is a union of n distinct conjugacy classes of G. Set $\eta(X) = n$.

We can check that given any two conjugacy classes A and B of G, the product $AB = \{ab \colon a \in A, b \in B\}$ of A and B is a G-invariant set. Thus $\eta(AB)$ is the number of distinct conjugacy classes of G such that AB is the union of those classes.

Denote by $\mathbf{C}_G(A) = \{g \in G \colon a^g = a \text{ for all } a \in A\}$ the centralizer of the set A in G. If G is a solvable group, denote by $\mathrm{dl}(G)$ the derived length of G. Let $\mathbf{Z}(G)$ be the center of G.

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In this note, we are exploring the relations between the structure of the group G and the product AB of some conjugacy classes A and B of G. More specifically, we are exploring the relation between the derived length of some section of G and properties of AB.

Given a finite solvable group G and conjugacy classes A and B of G, is there any relationship between the derived length of G and $\eta(AB)$? In general, the answer seems to be no. For instance, $A\{e\} = A$ for any finite group G and any conjugacy class A of G. Thus $\eta(AB)$ may not give us information about $\mathrm{dl}(G)$, but it does give us a linear bound on $\mathrm{dl}(G/\mathbf{C}_G(A))$ when $B = A^{-1}$ and G is supersolvable. More precisely

Theorem A: *For any finite supersolvable group G and any conjugacy class A of G we have that*

$$\mathrm{dl}(G/\mathbf{C}_G(A)) \leq 2\eta(AA^{-1}) - 1. \tag{1.1}$$

An application of this result is the following

Corollary B: *For any finite supersolvable group and any conjugacy classes A, B of G such that $AB \cap \mathbf{Z}(G) \neq \emptyset$, we have that*

$$\mathrm{dl}(G/\mathbf{C}_G(A)) \leq 2\eta(AB) - 1.$$

We want now to point out the "dual" situation with characters.

Denote by $\mathrm{Irr}(G)$ the set of irreducible complex characters of G. We can check that the product of characters is a character. Therefore, $\chi\psi$ is a character of G for any $\chi, \psi \in \mathrm{Irr}(G)$. It is known that a character can be expressed as an integral linear combination of irreducible characters. Then the decomposition of the character $\chi\psi$ into its distinct irreducible constituents $\theta_1, \theta_2, \ldots, \theta_n$ has the form

$$\chi\psi = \sum_{i=1}^{n} m_i\theta_i$$

where $n > 0$ and m_i is the multiplicity of θ_i. Set $\eta(\chi\psi) = n$, so that $\eta(\chi\psi)$ is the number of distinct irreducible constituents of the product $\chi\psi$. Define $\overline{\chi}(g)$ to be the complex conjugate $\overline{\chi(g)}$ of $\chi(g)$ for all $g \in G$.

In Theorem A of [1], it is proved that there exist constants c and d such that for any finite solvable group G and any irreducible character χ of G,

$$\mathrm{dl}(G/\operatorname{Ker}(\chi)) \leq c\eta(\chi\overline{\chi}) + d.$$

If, in addition, G is a supersolvable group, then we may take $c = 2$ and $d = -1$.

We regard $\mathbf{C}_G(A)$ for the conjugacy class A as the dual of $\mathrm{Ker}(\chi)$ for a character $\chi \in \mathrm{Irr}(G)$, the conjugacy class A^{-1} for A as the dual of $\overline{\chi}$ for the character $\chi \in \mathrm{Irr}(G)$. Thus we regard Theorem A as the dual in conjugacy classes of Theorem A of [1] for supersolvable groups. In light of Theorem A of [1], we wonder

Conjecture: *There exist universal constants q and r such that for any finite solvable group and any conjugacy class A of G, we have that*

$$\mathrm{dl}(G/\mathbf{C}_G(A)) \leq q\eta(AA^{-1}) + r.$$

We want to remark that there are several results showing the "duality" of products of conjugacy classes and products of characters. For example, see [8], [3] and [4], [6] and [5], [2] and [7]. However, we also want to remark that there are results in products of conjugacy classes that do not hold true for the "dual" in characters. For instance, it has been proved that the product of nontrivial conjugacy classes in A_n, for $n \geq 5$, is never a conjugacy classes (see [9]). On the other hand, if $n \geq 5$ is a perfect square, there exist irreducible characters χ and ψ in A_n such that the product $\chi\psi$ is also an irreducible (see [10]).

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2. The function η

Let G be a finite group. Observe that if A is a conjugacy class of G, then for any $a \in A$ we have that $A = a^G = \{a^g : g \in G\}$.

In this section, we show that given a subgroup H with $\mathbf{C}_G(a) \subseteq H \subseteq G$, we may not have a relation between $\eta(a^G(a^{-1})^G)$ and $\eta(a^H(a^{-1})^H)$.

Example 2.1: If H is a subgroup of G with $\mathbf{C}_G(a) \subset H \subseteq G$, then it is not necessary that $\eta(a^G(a^{-1})^G) \geq \eta(a^H(a^{-1})^H)$.

Proof. Fix a prime p. Let $\mathrm{GF}(p) = \{0, 1, \ldots, p-1\}$ be a finite field with p elements. Denote by F^* the group of units of $\mathrm{GF}(p)$. Also denote by F the additive group of $\mathrm{GF}(p)$. Observe that F^* acts on F by multiplication. Define M to be the semi-direct product $F \rtimes F^*$ of F by F^*.

Let $C = \langle c \rangle$ be a cyclic group of order p and 1_C be the identity of C. Let K be the direct product of p-copies of C, i.e

$$K = C \times \cdots \times C.$$

Thus K is an elementary abelian group of order p^p. Given any $a \in F$ and any $b \in F^*$, define the action of $(a,b) \in M$ in $\mathrm{GF}(p)$ as $x \mapsto bx + a$. With the previous action, we can then define an action of M in K by permuting the entries of the elements of K, i.e. given any $k \in K$, the action of $(a,b) \in M$ is that the i-th entry of k is the $(bi+a)$-th entry of the element $(a,b)^{-1}k(a,b) \in K$.

Let G be the wreath product of C and M relative to $\mathrm{GF}(p)$, i.e. $G = K \rtimes M$. Also, define $H = K \rtimes F$.

Set $a = (c, 1_C, \ldots, 1_C)$ in K. By Theorem A of [4], we have that $\eta(a^H(a^{-1})^H) = p$. By Proposition 5.4 of [4], we have that $\eta(a^G(a^{-1})^G) = 2$. ∎

Example 2.2: If H is a subgroup of G with $\mathbf{C}_G(a) \subseteq H \subset G$, then we need not have $\eta(a^G(a^{-1})^G) \leq \eta(a^H(a^{-1})^H)$.

Proof. Let G be an extra-special group of exponent p and order p^3, for some odd prime p. Let $a \in G \backslash \mathbf{Z}(G)$ and $H = \mathbf{C}_G(a)$. We can check that $\eta(a^H(a^{-1})^H) = 1$ and $\eta(a^G(a^{-1})^G) = p$. ∎

3. Proof of Theorem A

Hypotheses 3.1: *Let G be a solvable group, $N \trianglelefteq G$ be a normal subgroup of G, A be a conjugacy class of G. Fix $a \in A$. Set $C = \mathbf{C}_G(a)$. Since $a \in A$, observe that $a^G = A$ and $\mathbf{C}_G(A) = \mathrm{core}_G(C)$. Set $\bar{K} = (KN)/N$ for any subgroup of K of G, and $\bar{g} = gN$ for any element $g \in G$. Set $C_N = \{g \in G \colon [a,g] \in N\}$.*

Lemma 3.2: *Assume Hypothesis 3.1. Then the set C_N is a subgroup of G containing CN, $C_N/N = \mathbf{C}_{\bar{G}}(\bar{a})$, and*

$$\eta(\bar{a}^{\bar{G}}(\bar{a}^{-1})^{\bar{G}}) + \eta((a^{C_N}(a^{-1})^{C_N})^G) - 1 \leq \eta(a^G(a^{-1})^G). \tag{3.3}$$

Proof. Let $\rho : G \to G/N$ be the homomorphism defined by $\rho(g) = \bar{g}$. We can check that $\rho(g) = \bar{g} \in \mathbf{C}_{\bar{G}}(\bar{a})$ if and only if $[a,g] \in N$, that is if and only if $g \in C_N$, i.e. $\rho(a^g a^{-1}) = \bar{e}$ if and only if $g \in C_N$. Thus C_N is the inverse image of the group $\mathbf{C}_{\bar{G}}(\bar{a})$ under ρ.

Since $\rho(a^g a^{-1}) = \bar{e}$ if and only if $g \in C_N$, we have

$$(\bigcup_{g\in C_N} (a^g a^{-1})^G) \cap (\bigcup_{g\in G\setminus C_N} (a^g a^{-1})^G) = \emptyset,$$

and

$$\eta(\bar{a}^{\bar{G}}(\bar{a}^{-1})^{\bar{G}}) - 1 \leq \eta(\bigcup_{g\in G\setminus C_N} (a^g a^{-1})^G).$$

Observe that

$$a^G(a^{-1})^G = (\bigcup_{g\in C_N} (a^g a^{-1})^G) \cup (\bigcup_{g\in G\setminus C_N} (a^g a^{-1})^G).$$

Since $\cup_{g\in C_N}(a^g a^{-1})^G = (a^{C_N}(a^{-1})^{C_N})^G$, (3.3) follows. ∎

LEMMA 3.4: *Let H and K be subgroups of G with $K \subseteq H$. Then*

$$\mathrm{dl}(\mathrm{core}_G(H)/\,\mathrm{core}_G(K)) \leq \mathrm{dl}(H/\,\mathrm{core}_H(K)).$$

Proof. Since $\mathrm{core}_G(H) = \bigcap_{g\in G} H^g$, the map

$$\varphi : \mathrm{core}_G(H) \to \bigoplus_{g\in G} H^g/\,\mathrm{core}_{H^g}(K^g)$$

defined by $\varphi(h) = \bigoplus_{g\in G} h\,\mathrm{core}_{H^g}(K^g)$ is well-defined. Observe that the kernel of φ is

$$\{h \in \mathrm{core}_G(H)\colon h \in K^g \text{ for all } g \in G\} = \mathrm{core}_G(K).$$

Thus $\mathrm{core}_G(H)/\,\mathrm{core}_G(K)$ is isomorphic to a section of $\bigoplus_{g\in G} H^g/\,\mathrm{core}_{H^g}(K^g)$ and so

$$\mathrm{dl}(\mathrm{core}_G(H)/\,\mathrm{core}_G(K)) \leq \mathrm{dl}(\bigoplus_{g\in G} H^g/\,\mathrm{core}_{H^g}(K^g)).$$

Since $H^g/\,\mathrm{core}_{H^g}(K^g)$ is isomorphic to $H/\,\mathrm{core}_H(K)$, then

$$\mathrm{dl}(H/\,\mathrm{core}_H(K)) = \mathrm{dl}(\oplus_{g\in G} H^g/\,\mathrm{core}_{H^g}(K^g)),$$

and the result follows. ∎

LEMMA 3.5: *Assume Hypothesis 3.1. Assume also that N is abelian. Then*

$$\mathrm{dl}(C_N/\,\mathrm{core}_{C_N}(C)) \leq \mathrm{dl}(C_N/(C_N \cap \mathbf{C}_G(N))) + 1. \tag{3.6}$$

If, in addition, we have that N is a cyclic subgroup, then

$$\mathrm{dl}(C_N/\,\mathrm{core}_{C_N}(C)) \leq 2. \tag{3.7}$$

Proof. Since $C_N = \{g \in G\colon [a,g] \in N\}$, we have

$$a^{C_N}(a^{-1})^{C_N} \subseteq N. \tag{3.8}$$

Set $H = C_N \cap \mathbf{C}_G(N)$. Observe that $\mathbf{Z}(H) = \mathbf{Z}(C_N \cap \mathbf{C}_G(N)) \supseteq N$. By (3.8), it follows that $a \in \mathbf{Z}_2(H)$ and thus $\mathbf{C}_H(a) \supseteq [H,H]$. Since $H \trianglelefteq C_N$, $[H,H] \trianglelefteq C_N$. Since $[H,H] \trianglelefteq C_N$ and $\mathbf{C}_H(a) = C \cap H \supseteq [H,H]$, we have that $\operatorname{core}_{C_N}(C) \cap \mathbf{C}_G(N) = \operatorname{core}_{C_N}(C) \cap H \supseteq [H,H]$. Thus

$$\operatorname{dl}(H/(H \cap \operatorname{core}_{C_N}(C))) \leq 1. \tag{3.9}$$

Then

$$\begin{aligned} \operatorname{dl}(C_N/\operatorname{core}_{C_N}(C)) &\leq \operatorname{dl}(C_N/H) + \operatorname{dl}(H/(\operatorname{core}_{C_N}(C) \cap H)) \\ &\leq \operatorname{dl}(C_N/H) + 1, \end{aligned} \tag{3.10}$$

where the last inequality follows from (3.9).

If N is a cyclic group, then the group $G/\mathbf{C}_G(N)$ is abelian and so $C_N/H = C_N/(C_N \cap \mathbf{C}_G(N))$ is abelian. Thus (3.7) follows from (3.6) ∎

Lemma 3.11: *Let N be a normal subgroup of G and $a \in G$. Suppose that*

$$\eta(\bar{a}^{\bar{G}}(\bar{a}^{-1})^{\bar{G}}) = \eta(a^G(a^{-1})^G).$$

Then $N \subseteq \mathbf{C}_G(a) = C_N$. In particular

$$\operatorname{dl}(G/\operatorname{core}_G(\mathbf{C}_G(a))) = \operatorname{dl}(\bar{G}/\operatorname{core}_{\bar{G}}(\mathbf{C}_{\bar{G}}(\bar{a}))).$$

Proof. Since $\eta(\bar{a}^{\bar{G}}(\bar{a}^{-1})^{\bar{G}}) = \eta(a^G(a^{-1})^G)$, by Lemma 3.2 we have that

$$\eta((a^{C_N}(a^{-1})^{C_N})^G) = 1.$$

Since $e \in a^{C_N}(a^{-1})^{C_N}$ and $e^G = \{e\}$, it follows that $a^{C_N}(a^{-1})^{C_N} = \{e\}$ and so $C_N = \mathbf{C}_G(a)$. Since $C_N/N = \mathbf{C}_{\bar{G}}(\bar{a})$, the result follows. ∎

Proof of Theorem A. We are going to use induction on $\eta(AA^{-1})|G|$. Observe that the statement is true if $\eta(AA^{-1}) = 1$, since in that case $AA^{-1} = \{e\}$ and thus $\mathbf{C}_G(A) = G$.

We are going to use the notation of Hypothesis 3.1, where N, in addition, is a minimal normal subgroup of G. Thus $A = a^G$ and $\mathbf{C}_G(A) = \operatorname{core}_G(C)$. Lemma 3.2 implies that $\eta(\bar{a}^{\bar{G}}(\bar{a}^{-1})^{\bar{G}}) \leq \eta(a^G(a^{-1})^G)$. Since $|\bar{G}| < |G|$, by induction we have that

$$\operatorname{dl}(\bar{G}/\operatorname{core}_{\bar{G}}(\mathbf{C}_{\bar{G}}(\bar{a}))) \leq 2\eta(\bar{a}^{\bar{G}}(\bar{a}^{-1})^{\bar{G}}) - 1. \tag{3.12}$$

Observe that $\bar{G}/\operatorname{core}_{\bar{G}}(\mathbf{C}_{\bar{G}}(\bar{a}))$ is isomorphic to $G/\operatorname{core}(\mathbf{C}_G(C_N))$. Therefore

$$\mathrm{dl}(G/\operatorname{core}(\mathbf{C}_G(C_N)) = \mathrm{dl}(\bar{G}/\operatorname{core}_{\bar{G}}(\mathbf{C}_{\bar{G}}(\bar{a}))). \tag{3.13}$$

Assume that $\operatorname{core}_G(C_N) = \operatorname{core}_G(C)$. Then $G/\operatorname{core}_G(C)$ is isomorphic to the group $\bar{G}/\operatorname{core}_{\bar{G}}(\mathbf{C}_{\bar{G}}(\bar{a}))$. By (3.12), (3.13) and Lemma 3.2, we have

$$\begin{aligned}\mathrm{dl}(G/\operatorname{core}_G(C)) &= \mathrm{dl}(\bar{G}/\operatorname{core}_{\bar{G}}(\mathbf{C}_{\bar{C}}(\bar{a}))\\ &\le 2\eta(\bar{a}^{\bar{G}}(\bar{a}^{-1})^{\bar{G}}) - 1 \le 2\eta(a^G(a^{-1})^G) - 1.\end{aligned}$$

We may assume then that $\operatorname{core}_G(C)$ is a proper subset of $\operatorname{core}_G(C_N)$. Observe that then C is properly contained in C_N and therefore, by Lemma 3.11, we must have that

$$\eta(\bar{a}^{\bar{G}}(\bar{a}^{-1})^{\bar{G}}) < \eta(a^G(a^{-1})^G).$$

So $\eta(\bar{a}^{\bar{G}}(\bar{a}^{-1})^{\bar{G}}) + 1 \le \eta(a^G(a^{-1})^G)$. Since G is supersolvable, N is cyclic. By Lemma 3.4, (3.7), (3.12) and (3.13) we have that

$$\begin{aligned}\mathrm{dl}(G/\operatorname{core}_G(C)) &\le \mathrm{dl}(G/\operatorname{core}_G(C_N)) + \mathrm{dl}(\operatorname{core}_G(C_N)/\operatorname{core}_G(C))\\ &= \mathrm{dl}(\bar{G}/\operatorname{core}_{\bar{G}}(\mathbf{C}_{\bar{G}}(\bar{a}))) + \mathrm{dl}(\operatorname{core}_G(C_N)/\operatorname{core}_G(C))\\ &\le [2\eta(\bar{a}^{\bar{G}}(\bar{a}^{-1})^{\bar{G}}) - 1] + 2\\ &\le 2(\eta(a^G(a^{-1})^G) - 1) - 1 + 2\\ &\le 2\eta(a^G(a^{-1})^G) - 1.\end{aligned}$$

The proof is now complete. ∎

4. Proof of Corollary B

LEMMA 4.1: *Let G be a finite group, A and B be conjugacy classes of G such that $AB \cap \mathbf{Z}(G) \neq \emptyset$. Then $\eta(AB) = \eta(AA^{-1})$.*

Proof. Since $AB \cap \mathbf{Z}(G) \neq \emptyset$, there exist some $z \in \mathbf{Z}(G)$, $a \in A$ and $b \in B$ such that $z = ab$. Thus $b = a^{-1}z$ and so $AB = a^G b^G = a^G(a^{-1}z)^G = (a^G(a^{-1})^G)z = ABz$. It follows then that $\eta(AB) = \eta(AA^{-1})$. ∎

Corollary B follows from Lemma 4.1 and Theorem A.

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